The solutions of two conjectures for convex sets

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1.- Introduction : R. J. Gardner, S. Kwapien and D. P. Laurie working on a conjecture of B. Grünbaum, see [2] , reached two elegant conjectures, including inequalities, about ovals (compact plane convex sets).

The author gives proofs for these conjectures below.

The above mentioned conjectured the following, see [1] .

- . Let F be an oval and L_1, L_2, L_3 three concurrent str. lines through the same interior point of F dividing F into six regions with area A_i, B_i $i=1,2,3$, see fig. (1), below.

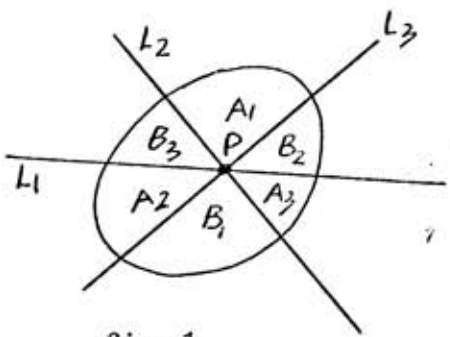

fig.1

First conjecture: $$\frac{A_1}{B_1} + \frac{A_2}{B_2} + \frac{A_3}{B_3} \geq \frac{3}{2}$$

Second conjecture: $$\frac{A_2+A_3}{B_1} + \frac{A_3+A_1}{B_2} + \frac{A_1+A_2}{B_3} \geq 3$$

2.- Proof of the first conjecture.

Theorem 1. $$\frac{A_1}{B_1}+\frac{A_2}{B_2}+\frac{A_3}{B_3}\geqslant\frac{3}{2}$$

Proof.

Let L_1 intersects F at the points L, G, L_2 at the points K, E and L_3 at the points D, H. The lines KL, HG, DE are being intersected at the points A, C, B.

CASE 1.

Suppose that P is an interior point of the triangle ABC see fig. (2).

It is obvious that:

$$\sum_{i=1}^{3}\frac{A_i}{B_i}\geqslant\frac{[PDE]}{[PKAH]}+\frac{[PGH]}{[PDBL]}+\frac{[PLK]}{[PGCE]} \qquad (1)$$

Where $[W]$ denotes the area of the figure W.

From (1) it is easily understood that, it is enough to prove the problem for a triangle. Taking into account the properties and the theorems of an affinity transformation, we can see that, there is an affinity transforming ABC into an equilateral triangle.

Also, we know that affinities preserve the area ratio. Therefore we only have to prove the problem for an equilateral triangle.

Let ABC be an equilateral triangle and P an interior point with x, y, z its distance from each of the sides BC, CA, AB. The parallel through P to the side BC intersects AB and AC at the points M, N respectively. (see fig.3)

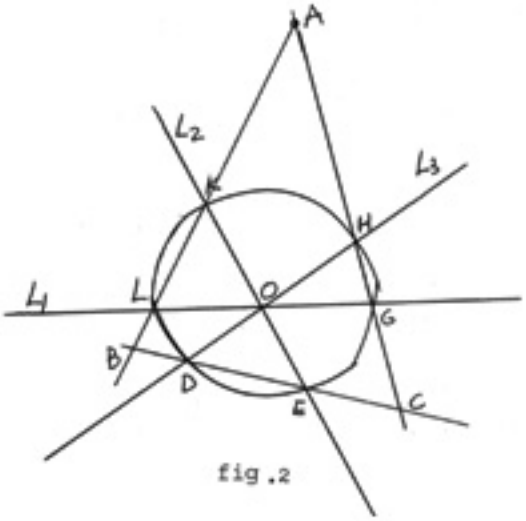

fig.2

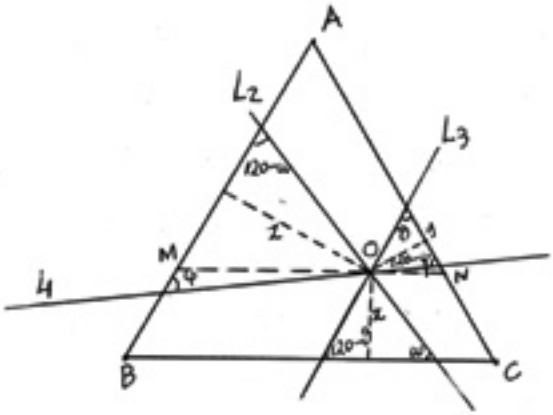

fig.3

It is elementary to calculate the areas of the triangles MKP, HPN . We find:

$$[MKP] = \frac{(MP)^2\sqrt{3}\,\sin\omega}{4\sin(120^\circ-\omega)} \qquad [HPN] = \frac{(PN)^2\sqrt{3}\,\sin(120^\circ-\vartheta)}{4\sin\vartheta}$$

Then using the inequality

$$\lambda_1 x_1^2+\lambda_2 x_2^2 \geqslant \frac{\lambda_1\lambda_2(x_1+x_2)^2}{\lambda_1+\lambda_2}, \quad \lambda_1,\lambda_2,x_1,x_2 \in \mathbb{R}, \quad \lambda_1+\lambda_2>0 \qquad (2$$

we have:

$$[MKP] + [HPN] = \lambda_1(MP)^2+\lambda_2(PN)^2 \geqslant \frac{\lambda_1\lambda_2(MN)^2}{\lambda_1+\lambda_2}$$

If we take $\lambda_1 = \dfrac{\sqrt{3}\,\sin\omega}{4\sin(120^\circ-\omega)}$, $\lambda_2 = \dfrac{\sqrt{3}\,\sin(120-\vartheta)}{4\sin\vartheta}$

then, an easy calculation will give.

$$[PKAH] = [AMN] - [MPK] - [HPN] \leqslant \frac{3(MN)^2}{8}\left[\frac{\sqrt{3}}{2}+\frac{1}{\operatorname{cotg}\omega+\operatorname{cotg}(120^\circ-\vartheta)}\right]^{-1}$$

$$(AMN) = \frac{MN^2\sqrt{3}}{4}$$

Also $\quad [PDE] = \frac{1}{2} x^2 [\cot g\, \omega + \cot g\,(120^\circ - \vartheta)]$

Therefore, easily we obtain that:

$$\frac{[PDE]}{[PKAH]} \geqslant \left(\frac{x}{y+z}\right)^2 \left[1 + \frac{\sqrt{3}}{2}\left(\cot g\, \omega + \cot g\,(120 - \vartheta)\right)\right]$$

Denoting $\quad \cot g\, \omega = p, \quad \cot g\, \vartheta = q, \quad \cot g\, \varphi = r$

we have to prove that:

$$\sum \left(\frac{x}{y+z}\right)^2 + \sum \frac{\sqrt{3}}{2}\left(\frac{x}{y+z}\right)^2 \left[p + \frac{\sqrt{3} - q}{1 + q\sqrt{3}}\right] \geqslant \frac{3}{2} \qquad (3)$$

where the sums are cyclic over $\quad x, y, z - p, q, r$

or the equivalent to (3)

$$\sum \left[\left(\frac{x}{y+z}\right)^2 (1 + p\sqrt{3}) + 4\left(\frac{z}{x+y}\right)^2 \frac{1}{1 + p\sqrt{3}}\right] \geqslant 3 \qquad (4)$$

but,

$$\left(\frac{x}{y+z}\right)^2 (1 + p\sqrt{3}) + 4\left(\frac{z}{x+y}\right)^2 \frac{1}{1 + p\sqrt{3}} \geqslant \frac{4xz}{(x+y)(y+z)}. \qquad (5)$$

So, we have to prove that

$$4\sum \frac{xz}{(x+y)(y+z)} \geqslant 3 \qquad (6)$$

or, that: $\quad \sum xy(x+y) \geqslant 6xyz$

which is elementary.

<u>The equality</u> follows from $(1),(2),(5),(6)$ that is the equality holds iff: $x=y=z$ and $p=q=r=\frac{1}{\sqrt{3}}$ or which is the same, if and only if F is a triangle and P coincides with the centroid and the lines L_1, L_2, L_3 are parallel to the sides respectively.

CASE 2.

Suppose that P is not an interior point of the triangle ABC, see fig. (4).

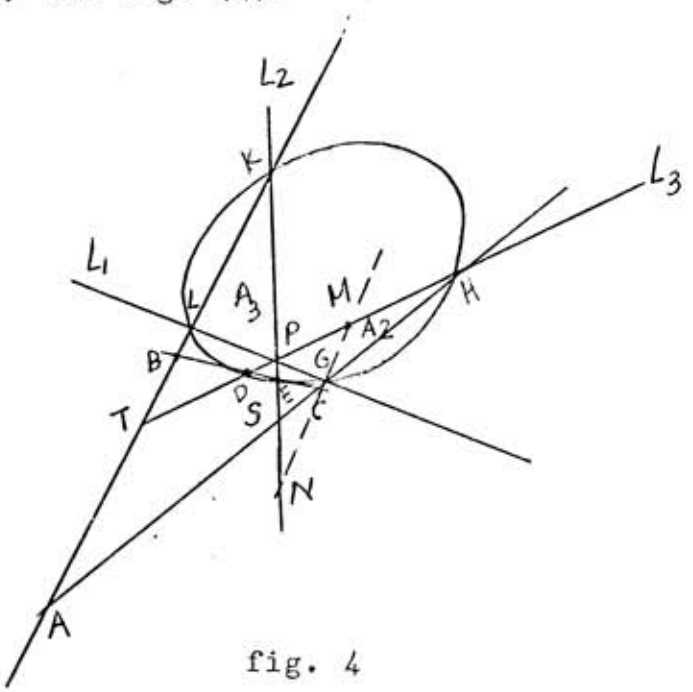

fig. 4

It is very easy to prove $\frac{A_2}{B_2}+\frac{A_3}{B_3}>\frac{3}{2}$

Because the obvious inequalities

$$\frac{A_2}{B_2}>\frac{A_2}{[LPT]}, \quad \frac{A_3}{B_3}>\frac{A_3}{[PGS]}$$

we have to prove

$$\frac{A_2}{[LPT]}+\frac{A_3}{[PGS]}>\frac{3}{2}$$

We take the str. line MGN parallel to KL, then

$$\frac{A_2}{[LPT]} > \frac{[PGM]}{[LPT]} = \frac{(PG)^2}{(PL)^2} \quad \text{, also} \quad \frac{A_3}{[PGS]} > \frac{A_3}{[PGN]} = \frac{(PL)^2}{(PG)^2}$$

Therefore
$$\frac{A_2}{[LPT]} + \frac{A_3}{[PGS]} > \frac{(PG)^2}{(PL)^2} + \frac{(PL)^2}{(PG)^2} \geqslant 2 > \frac{3}{2}$$

CASE 3.

Suppose KL parallel to HG. The proof as in the case 2.

3.- The proof of the second conjecture.

<u>Theorem 2.</u>
$$\frac{A_2+A_3}{B_1} + \frac{A_1+A_3}{B_2} + \frac{A_1+A_2}{B_3} \geqslant 3$$

Proof

CASE 1.

Suppose that P is an interior point of the triangle ABC see fig.(2)

As in the first conjecture, from the fig. (2), it is enough to prove

$$\frac{A_2+A_3}{B_1} + \frac{A_3+A_1}{B_2} + \frac{A_1+A_2}{B_3} \geqslant \frac{[PKL]+[PGH]}{[PKAH]} + \frac{[PKL]+[PDE]}{[PLBD]} + \frac{[PDE]+[PGH]}{[PECG]} \qquad (7)$$

We need the following lemma.

<u>Lemma</u>

Let ABC be a triangle and P, N, M are points on the sides BC, CA, AB respectively. We will show that:

$$Q_A = \frac{[BPM]+[PCN]}{[AMPN]} \geq \frac{\sin(B+\omega)\sin\varphi}{\sin B\sin(\varphi+\omega)} + \frac{\sin(C+\varphi)\sin\omega}{\sin C\sin(\varphi+\omega)} - 1$$

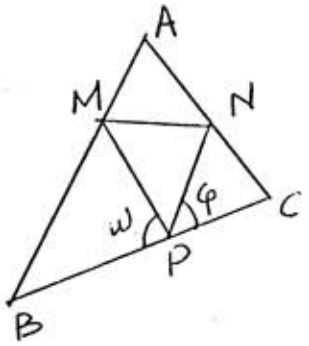

fig. 5

where A, B, C the angles of the triangle ABC

$\widehat{BPM} = \omega$, $\widehat{CPN} = \varphi$. The equality iff $MN \parallel BC$

It is elementary to see that:

$$\frac{[BPM]}{[ABC]} = \frac{BM.BP}{c.a} = \frac{K_1\, BP^2}{c.a}$$

$$\frac{[PCN]}{[ABC]} = \frac{CN.CP}{b.a} = \frac{K_2\, CP^2}{b.a}$$

where $K_1 = \dfrac{\sin\omega}{\sin(B+\omega)}$ $K_2 = \dfrac{\sin\varphi}{\sin(C+\varphi)}$

Using the well known inequality (2)

$$\lambda_1 x_1^2 + \lambda_2 x_2^2 \geq \frac{\lambda_1\lambda_2(x_1+x_2)^2}{x_1+x_2}$$

we obtain

$$\frac{[BPM]+[PCN]}{[ABC]} \geq \frac{aK_1K_2}{bK_1+cK_2}$$

therefore it holds:

$$Q_A = \frac{1}{\frac{[ABC]}{[BMP]+[PCN]} - 1} \geqslant \frac{1}{\frac{bk_1+ck_2}{ak_1k_2} - 1} \tag{8}$$

but $$\frac{bk_1+ck_2}{ak_1k_2} = \frac{\sin B \sin(C+\varphi)}{\sin A \sin\varphi} + \frac{\sin C \sin(B+\omega)}{\sin A \sin\omega} =$$

$$= \frac{\sin B \sin C}{\sin A}\left[\operatorname{cotg}\varphi + \operatorname{cotg}\omega\right] + 1 \tag{9}$$

From (8) and (9) follows:

$$Q_A \geq \frac{\sin A}{\sin B \sin C\left[\operatorname{cotg}\omega + \operatorname{cotg}\varphi\right]} \tag{10}$$

It is very easy and elementary to see that the following identity holds

$$\frac{\sin A}{\sin B \sin C\left[\operatorname{cotg}\omega + \operatorname{cotg}\varphi\right]} = \frac{\sin(B+\omega)\sin\varphi}{\sin B \sin(\varphi+\omega)} + \frac{\sin(C+\varphi)\sin\omega}{\sin C \sin(\varphi+\omega)} - 1 \tag{11}$$

(10) and (11) prove our lemma.

The equality from (2), when

$$\lambda_1 x_1 = \lambda_2 x_2$$

or $\dfrac{K_1 BP}{c} = \dfrac{K_2 CP}{b}$ or $\dfrac{BM}{c} = \dfrac{CN}{b}$

that is MN parallel to BC.

We are now ready to prove the second conjecture.

Let,

$$Q = \frac{[KLP]+[PGH]}{[AKPH]} + \frac{[KLP]+[DEP]}{[BDPL]} + \frac{[DEP]+[PGH]}{[PEGC]}$$

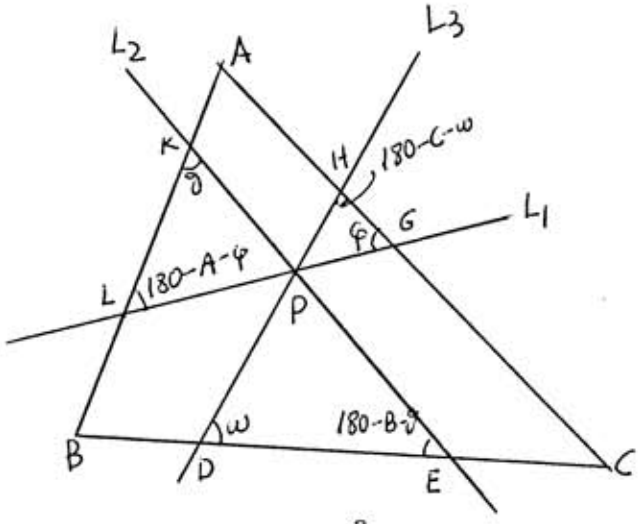

fig. 5[a]

$$Q_A = \frac{[KLP]+[PGH]}{[AKPH]} \geqslant \frac{\sin\vartheta \sin(180-C-\omega+\varphi)}{\sin(180-A-\varphi)\sin(180-B+\vartheta+\omega)} + \frac{\sin(180-C-\omega)\sin(180-A-\varphi+\vartheta)}{\sin\varphi \sin(180-B-\vartheta+\omega)} - 1$$

Therefore $Q = \sum\limits_{A,B,C} Q_A$ or

$$Q = \sum_{\substack{\omega,\varphi,\vartheta \\ A,B,C}} \left(\frac{\sin\vartheta \sin(180^\circ - C-\omega+\varphi)}{\sin(180-A-\varphi)\sin(180-B-\vartheta+\omega)} + \frac{\sin(180-A-\varphi)\sin(180-B-\vartheta+\omega)}{\sin\vartheta \sin(180-C-\omega+\varphi)} \right) - 3 \qquad (12)$$

The sums are cyclic over ω, φ, γ and A, B, C
Obviously, follows

$$Q \geqslant 2+2+2-3=3$$

<u>The equality</u> according to our lemma if $KH \parallel LG$, $GE \parallel DH$ and $LD \parallel KE$. Also taking into account (12) we find that LG, DH, KE must be parallel to the sides of ABC, that is P must coincide with the cendroid and L_1, L_2, L_3 must be parallel to the sides. Therefore the equality holds iff F is a triangle and P its centroid, L_1, L_2, L_3 must be parallel to the sides respectively.

CASE 2.

Let $(\widehat{LK, GH}) = \partial \to 0$ for the triangle ABC holds:

$$\frac{[PKL]+[PGH]}{[AKPH]} + \frac{[PDE]+[PKL]}{[PLBD]} + \frac{[PDE]+[PGH]}{[PECG]} > 3$$

But

$$\frac{[PKL]+[PGH]}{[AKPH]} \to 0$$

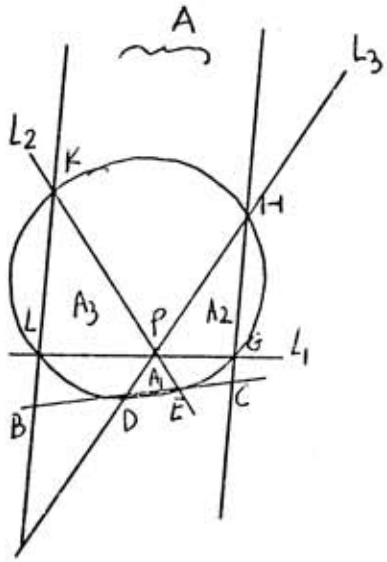

fig. 6

Taking into account the above we can easily prove the CASE 3, when P is an exterior point of ABC. (We take the parallel through L to GH).

Comment. The most part of Grümbaum's conjecture, see [2], follows as a natural consequense from theorem 1. The author of the above article has an easy Geometric proof for the case $f(K)=1$ which intends to publish in a separate paper.

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References

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2. B. Grünbaum, *Measures of symmetry for convex sets*, Proceedings of Symposia in Pure Math. Vol. VII, Convexity. Amer. Math. Soc. p. 270.
3. J. N. Lillington, *Problems related to a conjecture by Grünbaum*, Mathematika **21** (1974), pp. 45–54.

P.S. A different proof has been published by the author in Canad. Math. Bull. Vol 28(1), 1985, p.p. 60-66.